# 7.6 Solve Exponential and Logarithmic Equations

**Before** You studied exponential and logarithmic functions.

**Now** You will solve exponential and logarithmic equations.

**Why?** So you can solve problems about astronomy, as in Example 7.

**Key Vocabulary**
- exponential equation
- logarithmic equation
- extraneous solution, *p. 52*

**Exponential equations** are equations in which variable expressions occur as exponents. The result below is useful for solving certain exponential equations.

**KEY CONCEPT** *For Your Notebook*

**Property of Equality for Exponential Equations**

**Algebra** If $b$ is a positive number other than 1, then $b^x = b^y$ if and only if $x = y$.

**Example** If $3^x = 3^5$, then $x = 5$. If $x = 5$, then $3^x = 3^5$.

## EXAMPLE 1 Solve by equating exponents

Solve $4^x = \left(\frac{1}{2}\right)^{x-3}$.

| | |
|---|---|
| $4^x = \left(\frac{1}{2}\right)^{x-3}$ | Write original equation. |
| $(2^2)^x = (2^{-1})^{x-3}$ | Rewrite 4 and $\frac{1}{2}$ as powers with base 2. |
| $2^{2x} = 2^{-x+3}$ | Power of a power property |
| $2x = -x + 3$ | Property of equality for exponential equations |
| $x = 1$ | Solve for $x$. |

▸ The solution is 1.

**CHECK** Check the solution by substituting it into the original equation.

| | |
|---|---|
| $4^1 \stackrel{?}{=} \left(\frac{1}{2}\right)^{1-3}$ | Substitute 1 for $x$. |
| $4 \stackrel{?}{=} \left(\frac{1}{2}\right)^{-2}$ | Simplify. |
| $4 = 4$ ✓ | Solution checks. |

**GUIDED PRACTICE** for Example 1

**Solve the equation.**

1. $9^{2x} = 27^{x-1}$ $-3$
2. $100^{7x+1} = 1000^{3x-2}$ $-\frac{8}{5}$
3. $81^{3-x} = \left(\frac{1}{3}\right)^{5x-6}$ $-6$



## 1 PLAN AND PREPARE

### Warm-Up Exercises

**Transparency Available**

1. Write $\log_3(2x - 7) = 4$ in exponential form. $3^4 = 2x - 7$
2. Write $8^x = 30$ in logarithmic form. $\log_8 30 = x$

Solve the equation.

3. $100^x = 1000$ $\frac{3}{2}$
4. $\log_5 x = -3$ $\frac{1}{125}$
5. $x^2 - 7x - 60 = 0$ $-5, 12$

### Notetaking Guide

**Transparency Available**

Promotes interactive learning and notetaking skills, pp. 202–205.

### Pacing

**Basic:** 2 days
**Average:** 2 days
**Advanced:** 2 days
**Block:** 1 block
• See Teaching Guide/Lesson Plan.

## 2 FOCUS AND MOTIVATE

### Essential Question

**Big Idea 2, p. 477**

Why do logarithmic equations sometimes have extraneous solutions? **Tell students they will learn how to answer this question by solving logarithmic equations and checking the apparent solutions.**

***NCTM STANDARDS***

**Standard 2:** Analyze situations using algebraic symbols

**Standard 6:** Build knowledge through problem solving

## Resource Planning Guide

**Chapter Resource Book**
- Teaching Guide/Lesson Plan (pp. 62–63)
- Practice levels A, B, C (pp. 65–67)
- Study Guide (pp. 68–69)
- Catch-up for Absent Students (p. 70)
- Application (p. 71)
- Challenge (p. 72)

**Workbooks**
- Notetaking Guide (pp. 202–205)
- Practice Workbook (pp. 118–119)

**Teaching Options**
- **Power Presentations CD-ROM** provides dynamic electronic teaching resources for the classroom.
- **Activity Generator CD-ROM** provides editable activities for all ability levels.

**Interactive Technology**
- Easy Planner
- Power Presentations CD-ROM
- Activity Generator CD-ROM
- Animated Algebra
- Test Generator CD-ROM
- Online Quiz
- eWorkbook
- eEdition
- @HomeTutor

**Resources for English Learners**
- Quick Reference for English Learners
- Spanish Study Guide
- Multi-Language Visual Glossary
- Student Resources in Spanish

See also the Algebra 2 Toolkit for more strategies for meeting individual needs.

## Motivating the Lesson

Ask students who have studied radioactive decay in their science classes to explain the concept of half-life. Point out that the half-life of a radioactive substance can be found by using an exponential model with the natural base e.

# 3 TEACH

## Extra Example 1

Solve $3^x = \left(\frac{1}{9}\right)^{x+3}$. **−2**

### *Key Question to Ask for Example 1*

- For the method used to solve the equation to work, what must be true? **You must be able to write both sides as powers of the same base.**

## Extra Example 2

Solve $9^x = 35$. **about 1.62**

### *Key Question to Ask for Example 2*

- Why is it not possible to use the method of Example 1 in Example 2? **It is impossible to write 4 and 11 as powers of the same base.**

## Extra Example 3

Hot chocolate that has been heated to 90°C is poured into a mug and placed on a table in a room with a temperature of 20°C. If r = 0.145 when the time t is measured in minutes, how long will it take for the hot chocolate to cool to a temperature of 30°C? **about 13 min**

When it is not convenient to write each side of an exponential equation using the same base, you can solve the equation by taking a logarithm of each side.

**ANOTHER WAY**
For an alternative method for solving the problem in Example 2, turn to page 523 for the **Problem Solving Workshop**.

## EXAMPLE 2 Take a logarithm of each side

**Solve $4^x = 11$.**

| | |
|---|---|
| $4^x = 11$ | Write original equation. |
| $\log_4 4^x = \log_4 11$ | Take $\log_4$ of each side. |
| $x = \log_4 11$ | $\log_b b^x = x$ |
| $x = \dfrac{\log 11}{\log 4}$ | Change-of-base formula |
| $x \approx 1.73$ | Use a calculator. |

▶ The solution is about 1.73. Check this in the original equation.

**NEWTON'S LAW OF COOLING** An important application of exponential equations is *Newton's law of cooling*. This law states that for a cooling substance with initial temperature $T_0$, the temperature $T$ after $t$ minutes can be modeled by

$$T = (T_0 - T_R)e^{-rt} + T_R$$

where $T_R$ is the surrounding temperature and $r$ is the substance's cooling rate.

## EXAMPLE 3 Use an exponential model

**CARS** You are driving on a hot day when your car overheats and stops running. It overheats at 280°F and can be driven again at 230°F. If $r = 0.0048$ and it is 80°F outside, how long (in minutes) do you have to wait until you can continue driving?

**Solution**

| | |
|---|---|
| $T = (T_0 - T_R)e^{-rt} + T_R$ | Newton's law of cooling |
| $230 = (280 - 80)e^{-0.0048t} + 80$ | Substitute for $T$, $T_0$, $T_R$, and $r$. |
| $150 = 200e^{-0.0048t}$ | Subtract 80 from each side. |
| $0.75 = e^{-0.0048t}$ | Divide each side by 200. |
| $\ln 0.75 = \ln e^{-0.0048t}$ | Take natural log of each side. |
| $-0.2877 \approx -0.0048t$ | $\ln e^x = \log_e e^x = x$ |
| $60 \approx t$ | Divide each side by −0.0048. |

▶ You have to wait about 60 minutes until you can continue driving.

## ✓ GUIDED PRACTICE for Examples 2 and 3

**Solve the equation.**

**4.** $2^x = 5$ **about 2.32**

**5.** $7^{9x} = 15$ **about 0.155**

**6.** $4e^{-0.3x} - 7 = 13$ **about −5.365**

**SOLVING LOGARITHMIC EQUATIONS** **Logarithmic equations** are equations that involve logarithms of variable expressions. You can use the following property to solve some types of logarithmic equations.

**KEY CONCEPT** *For Your Notebook*

**Property of Equality for Logarithmic Equations**

**Algebra** If $b$, $x$, and $y$ are positive numbers with $b \neq 1$, then $\log_b x = \log_b y$ if and only if $x = y$.

**Example** If $\log_2 x = \log_2 7$, then $x = 7$. If $x = 7$, then $\log_2 x = \log_2 7$.

## EXAMPLE 4 Solve a logarithmic equation

**Solve $\log_5 (4x - 7) = \log_5 (x + 5)$.**

$\log_5 (4x - 7) = \log_5 (x + 5)$ **Write original equation.**

$4x - 7 = x + 5$ **Property of equality for logarithmic equations**

$3x - 7 = 5$ **Subtract $x$ from each side.**

$3x = 12$ **Add 7 to each side.**

$x = 4$ **Divide each side by 3.**

▶ The solution is 4.

**CHECK** Check the solution by substituting it into the original equation.

$\log_5 (4x - 7) = \log_5 (x + 5)$ **Write original equation.**

$\log_5 (4 \cdot 4 - 7) \stackrel{?}{=} \log_5 (4 + 5)$ **Substitute 4 for $x$.**

$\log_5 9 = \log_5 9$ ✓ **Solution checks.**

**EXPONENTIATING TO SOLVE EQUATIONS** The property of equality for exponential equations on page 515 implies that if you are given an equation $x = y$, then you can *exponentiate* each side to obtain an equation of the form $b^x = b^y$. This technique is useful for solving some logarithmic equations.

## EXAMPLE 5 Exponentiate each side of an equation

**Solve $\log_4 (5x - 1) = 3$.**

$\log_4 (5x - 1) = 3$ **Write original equation.**

$4^{\log_4 (5x - 1)} = 4^3$ **Exponentiate each side using base 4.**

$5x - 1 = 64$ **$b^{\log_b x} = x$**

$5x = 65$ **Add 1 to each side.**

$x = 13$ **Divide each side by 5.**

▶ The solution is 13.

**CHECK** $\log_4 (5x - 1) = \log_4 (5 \cdot 13 - 1) = \log_4 64$

Because $4^3 = 64$, $\log_4 64 = 3$. ✓



## Extra Example 4

Solve $\log_4 (2x + 8) = \log_4 (6x - 12)$. **5**

## *Key Question to Ask for Example 4*

- If each side of an equation contains a single logarithm and there are no other terms, what must be true in order for you to be able to solve the equation by the method used in Example 4? **Both logarithms must have the same base.**

## Extra Example 5

Solve $\log_7 (3x - 2) = 2$. **17**

## Mathematical Reasoning

**Multiple Representations** The Key Concept box states the property of equality for logarithmic equations algebraically and gives an example. Ask students to state this property in words, using vocabulary such as logarithm and base, but without using any mathematical symbols. Then ask them to demonstrate this property graphically, using either a hand-drawn or calculator-generated graph.

## *Differentiated Instruction*

**Advanced** Have students work in a small group to research one-to-one functions. Ask these students to present a mini-lesson on this topic to the class. The presentation should discuss the connection between one-to-one functions and relevant concepts that have been discussed in Chapters 6 and 7: inverse functions, the horizontal line test, the relationship between exponential and logarithmic functions with the same base, the property of equality for exponential functions, and the property of equality for logarithmic functions.

See also the Algebra 2 Toolkit for more strategies.